Question 3. Consider the set of all solutions to the system of equations

$$x + 2y + 3z + u = 4$$
$$5x - 3y + 2z - 6u = 0$$

as a subset of $\mathbb{R}^4$. Do the following:

(a) Explain how you can or could determine that this solution set defines a plane in $\mathbb{R}^4$. Include in this explanation what the solution set may look like if it is not a plane in $\mathbb{R}^4$.

Before we start, consider the following system of equations (of three variables).

$$Ax + By + Cz = D \quad \cdots \text{①}$$
$$A'x + B'y + C'z = D' \quad \cdots \text{②}$$

Each equation is associated to a plane in $\mathbb{R}^3$, and

a point (x, y, z) is a solution to the system
$\Leftrightarrow$ (x, y, z) satisfies both equations ① & ②
$\Leftrightarrow$ (x, y, z) lies in both planes
$\Leftrightarrow$ (x, y, z) lies in the intersection of two planes.

Hence, we see that the solution set of this system is exactly the intersection of two planes defined by equations ① and ②. If the normal vectors $\langle A, B, C\rangle$ and $\langle A', B', C'\rangle$ are pointing in the same direction, then the planes either coincide or are parallel to each other; otherwise, the planes intersect in a line.

Now let us consider the given system of equations

$$x + 2y + 3z + u = 4 \quad \cdots \text{①}$$
$$5x - 3y + 2z - 6u = 0 \quad \cdots \text{②}$$

Each equation is associated to a hyperplane in $\mathbb{R}^4$. Note that a hyperplane is a subspace whose dimension is one less than that of its ambient space. In our case, hyperplanes have dimension three. As before, the solution set of the system can be identified with the intersection of two hyperplanes defined by equations ① & ② respectively.

Again a vector normal to each hyperplane can be given by the coefficients of the corresponding equation. Since the vectors $\langle 1, 2, 3, 1 \rangle$ and $\langle 5, -3, 2, -6 \rangle$ are pointing in different directions, the hyperplanes should have nontrivial intersection, a plane* in $\mathbb{R}^4$. If the normal vectors were parallel, then the hyperplanes would either coincide or have no intersection (depending on the constant terms).

* Let a and b be arbitrarily given real numbers. If (a, b, z, u) is a solution to above system, then z, u should satisfy the following equations:

$$3z + u = 4 - a - 2b$$
$$2z - 6u = -5a + 3b$$

This system of equations has a unique solution, and z and u can be expressed in terms of a and b. In other words, z and u are dependant on the values of x and y. We have two free variables x and y to describe our solution set, so we can guess that our solution set has dimension 2.